International Journal of Learning, Teaching and Educational Research
Vol. 20, No. 6, pp. 38-65, June 2021
https://doi.org/10.26803/ijlter.20.6.3

Impulsing the Development of Students' Competency Related to Mathematical Thinking and Reasoning through Teaching Straight-Line Equations

Bui Phuong Uyen
Can Tho University, Can Tho City, Vietnam
https://orcid.org/0000-0002-8098-3398

Lu Kim Ngan
Master Student, Can Tho University, Can Tho City, Vietnam
https://orcid.org/0000-0002-0737-8095

Nguyen Phuong Thao
An Giang University, VNU-HCM, Ho Chi Minh City, Vietnam
https://orcid.org/0000-0003-1213-2492

Duong Huu Tong
Can Tho University, Can Tho City, Vietnam
https://orcid.org/0000-0002-4522-9760

Abstract. The research was carried out to develop students' ability to think and reason mathematically by teaching straight-line equations in a plane. Accordingly, teaching activities were designed according to five learning stages, which were integrated with mathematical thinking levels according to Van Hiele's model. Simultaneously, the learners' mathematical thinking and reasoning competencies were assessed according to the competency requirements specified in the Mathematics General Education Program and the levels of Van Hiele's model, the above three aspects of knowledge, skills and attitudes. The experiment involved 84 students in class 10, 44 of whom were in the experimental group, and 40 were in the control group. The research results showed that students in the experimental group achieved higher mathematical thinking and reasoning skills. Specifically, the two groups had equivalent results for the level of visualization and analysis. However, at the informal deduction and formal deduction and rigor levels, the ranking results of the two groups had a clear difference. The study group observations and students' opinion surveys also revealed that learning stages were designed according to Van Hiele's model and thought-provoking measures and visual images and language contributed to students' interest in learning and positive thinking.

Keywords: mathematical thinking; reasoning; straight-line equations; coordinate geometry; Van Hiele's model

1. Introduction

Recent changes in society show that mathematics has evolved to meet the needs of society's demands. People have gained a lot from applying mathematics in real life, especially knowledge and math skills. It allows math education to contribute to workforce training for society, in which the mathematical competency consists of eight elements and is divided into two groups. The first group includes mathematical thinking, problem-solving, modeling and mathematical reasoning. The second group is concerned with using language and mathematical media, including mathematical representation, mathematical communication, symbols, tools, and media (Niss & Højgaard, 2011). Polya (1963) once wrote: "First and foremost, mathematics education must teach students to think" (p. 605), as did mathematician - Cartesian philosopher Descartes: I am, I exist (1641). Hence, in teaching mathematics, instead of teaching learners how to resolve a test, the teacher should promote the types of intellectual tasks that they will perform when applying the subject to personal problems or careers in many aspects of life (Paul & Elder, 2008). In other words, teachers need to help students form and improve their ability to think and reason mathematically - one of the keys to students achieving personal success and for seeing and constructing the world.

Mathematical thinking is a process in which at least one of the cognitive activities associated with mathematics occurs. Indeed, they may be reasoning, abstraction, judgment, presentation and transition between representations, visualization, deductive, inductive, analysis, synthesis, relation, generalization and proof (Tall, 2002). Moreover, mathematical reasoning is the process of forming and understanding mathematical ideas and concepts associated with available premises to make assertions and lead to conclusions (Mukono, 2015; Mumu & Tanujaya, 2019). Karadag (2009) considers reasoning to be analyzing problems to find solutions to problems through inductive or deduction, to develop and verify statements. Thus, the mathematical argument appears in the problem-solving process (Yildirim, 2000 as cited in Gunhan, 2014) and is the manifestation of higher mathematical thinking (Kenney & Lindquist, 2000 as cited in Gunhan, 2014), and expresses the ability to compute and present problems, to explain and adjust solutions or arguments (Kilpatrick et al., 2001 as cited in Gunhan, 2014).

In the General Education Program of Mathematics of Vietnam issued in 2018, students' ability to think and reason mathematically is shown through (1) performing thinking manipulations such as comparison, analysis, synthesis, specialization, generalization, analogy, inductive and deduction; (2) showing evidence, arguments and knowing reasonable arguments before concluding; and (3) explaining or adapting the problem-solving approach mathematically (Ministry of Education and Training [MoET], 2018). Moreover, the Organization for Economic Cooperation and Development (OECD, 2018) gives the manifestations of mathematical reasoning capability in the evaluation criteria of PISA 2021, including assessing situations, choosing problem-solving strategies, making combined conclusions, managing and describing options, and understanding how to apply them.

According to Niss and Højgaard (2011), Drijvers (2015), Sinwell (2017), OECD

(2018) and Drijvers et al. (2019), mathematical thinking and reasoning competencies are closely related to problem-solving and modeling capacity. According to the OECD (2018), mathematical arguments influence one's ability to deal with real-world problems, especially evaluating social problems. Mathematical reasoning contributes to developing other critical skills for the twenty-first century (OECD, 2018).

Many teaching methods have been applied to improve teaching effectiveness and speed up students' mathematical reasoning and thinking abilities. Research by Hudson et al. (2015) pointed out that the physical, intellectual and emotional factors are necessary for mathematical thinking to take place; at the same time, concerning the basic question and answer processes, mathematical thinking occurs in specialization and generalization processes to lead to judgment and affirmation processes. With this, mathematical thinking can be encouraged through comparison training, which pressures, conflicts, and provokes; it can be supported through the question, challenge, and reflection (Mason et al., 2010).

Teaching approaches such as realistic mathematics education (RME) are also studied to improve students' thinking and reasoning abilities. Research by Papadakis et al. (2016), Dhayanti et al. (2018) and Drijvers et al. (2019) reveals that applying RME in math teaching contributes to students' mathematical thinking ability. In addition, Soares's (2012) research on competency-based education also provides the theoretical basis of competency-based teaching models to develop learners' competencies with output-based approaches. Conversely, information technology also supports teachers in utilizing various teaching resources. Research by Cesaria and Herman (2012), Dhayanti et al. (2018) and Kovacevic (2019) emphasizes the positive impact of IT achievements such as ICT, Sketchpad math software and e-learning in teaching to enhance students' mathematical reasoning competencies.

Due to the world's increasing demand for thinking and reasoning, institutions and teachers have become critical. Accordingly, Hudson et al. (2015), OECD (2018) and Drijvers et al. (2019) believe that it is necessary to renew the curriculum of mathematics in the direction of enhancing mathematical reasoning and thinking abilities for students. At the same time, the Vietnamese Ministry of Education and Training has approved the General Education Program in Mathematics (2018) to promote mathematical competencies for students. The capacity to think and reason mathematically is one of the core competency components. Additionally, it is impossible not to mention the role of teachers in realizing these educational goals. According to Cesaria and Herman (2019), students' ability to reason mathematically is impulsed through activities and depends on the teacher's creativity in the learning process. Research by Hudson et al. (2015) conducted with primary school teachers indicated teachers' confidence, competencies, attitudes, and beliefs about mathematics and teachers' expectations and experiences impulse students' mathematical thinking skills.

Evaluating the ability to think and reason mathematically means that various assessment tools and methods, such as questioning and problem-solving exercises, can be used. Analysis, synthesis and systematization of knowledge must apply mathematical knowledge to explain and reason (MoET, 2018). Niss and Højgaard (2011) also provide some methods and tools for assessing mathematical reasoning and thinking ability, such as exercises or essay problems; speaking exercises, short questions and interviews; and essay projects. Further, the authors evidence the results of student work observations, study logs, portfolios, and forms of communication through writing, presentations, posters and products. Other media may also assess students' thinking abilities (Niss & Højgaard, 2011).

Educators must build scales of students' mathematical reasoning and thinking levels and suitable evaluation criteria to effectively teach and evaluate learners' ability to think and reason mathematically. The two educators, Prigourierre van Hiele and Dina van Hiele - Geldof produced the Van Hiele theory of teaching geometry, which can then be applied to teaching most topics in mathematics. Van Hiele's theory consists of three components: concepts, levels of thinking and learning stages (Land, 1990). According to Van Hiele's theory, the geometrical thinking levels are arranged from 0 to 4, including 0 - Visualization, 1 - Analysis, 2 – Informal deduction, 3 - Formal deduction and 4 - Rigor (Van Hiele, 1986). Additionally, Teppo (1991) argues that, according to Van Hiele's theory, the students' development from one level to the next results from intentional teaching organized in five stages of information, oriented, interpreted, free-oriented and integrated. Many studies have analyzed Van Hiele's model (Bell, 1983; Hoffer, 1983; Land, 1990; Teppo, 1991; Van Hiele, 1986), and it turns out that Van Hiele's model can be used in many areas of mathematics, including geometry and algebra (Bell, 1983; Land, 1990; Van Hiele, 1986). The studies from Vojkuvkova (2012) and Masilo (2018) show the effectiveness of integrating thinking levels with teaching phases according to the Van Hiele model into geometry teaching.

Much research has been done on teaching to speed up students' mathematical reasoning and thinking abilities using Van Hiele's model. Cresswell and Speelman (2020) put a sample of students and professors through a test of 11 logical reasoning problems selected from psychological studies. Research results have documented that the higher the math training is for the student, the correct number of problems. As a result, math instruction helps students improve their ability to reason and think logically. Gunhan's research (2014) aims to evaluate students' reasoning skills in geometry. This investigation explores how various qualitative research approaches may be used. When solving a problem, students are asked to speak aloud while they think. The collected data indicate that students have vastly different argument-making processes. Concerning Van Hiele's model of mathematical thinking levels, Vojkuvkova (2012) presented a theoretical study on this model and emphasized the possibility of applying Van Hiele's theory to mathematical communication. The results confirm that Van Hiele's hierarchy of levels helps students see better and comprehend learning more efficiently. In addition, there is also the theoretical research of Crowley (1987) on Van Hiele's model in the development of geometric thinking and the research of Gutierrez and Jaime (1998) on the assessment of the mathematical reasoning level of the issue. Research by Salifu et al. (2018) on the geometric thinking levels of pedagogical students also gives notable results. Although there are many studies on the Van Hiele model, there is no work showing the trio relationship, including the Van Hiele model, mathematical thinking and reasoning skills, and straight-line equations in the coordinate plane.

In the General Education Program of Mathematics of Vietnam released in 2018, the geometry section provides knowledge and skills at the level of logical reasoning, algebraic methods in geometry (vectors, coordinates), which helps develop imagination and connects with many real-world problems (MoET, 2018). From there, researchers can see the ability through geometry to realize the goal of forming and improving students' mathematical reasoning and thinking skills. Additionally, the equation of the line in two dimensions is a fundamental topic in the high school mathematics program. Students can better deal with plane transformations, tangent curves, and coordinate geometry if they know this relationship. Furthermore, this represents an important turning point in applying the algebraic method in the study of geometry. However, in practice, many students have difficulty learning the contents of straight-line equations because they are not used to this teaching method. For this reason, students must understand the content of the straight-line equations so that their capacity for thinking and mathematical reasoning can be developed.

2. Theoretical framework

The selection of a theoretical foundation is required before any research can be carried out successfully. Among the topics covered in this theoretical framework are mathematical thinking and reasoning, Van Hiele's model of mathematical thinking and reasoning levels, and Van Hiele's model of learning stages.

2.1 Mathematical thinking and reasoning in the context of Vietnam

The General Education Program of Mathematics of Vietnam indicates that mathematics contributes to the formation and development of students' mathematical competencies, including the following components: mathematical thinking and reasoning ability; mathematical modeling ability; ability to solve mathematical problems; mathematical communication competence; ability to use tools and means of learning mathematics. In particular, the specific expression of mathematical thinking and reasoning capacity and requirements for high school education are shown as follows:
(1) Perform mental operations relatively proficiently, especially detect similarities and differences in relatively complex situations and interpret the results of observations.
(2) Use reasoning, induction, and deductive methods to see different ways of solving problems.
(3) Ask and answer questions when reasoning and problem-solving. Explain, prove, adjust the solution performed mathematically.

2.2 Van Hiele's model of mathematical thinking and reasoning levels

The coordinate method in the plane combines geometric and algebraic factors using algebraic methods to overcome problems in geometry. Many studies show the effectiveness of assessing students' level of geometric thinking based on Van Hiele's model (Abdullah & Zakaria, 2013; Alex & Mammen, 2018; Feza & Webb, 2005).

Hence, to apply Van Hiele's model to this topic, it is necessary to combine models in geometry (Van Hiele, 1986) and algebra (Bell, 1983). Regardless of whether the levels are called differently, they are all ordered in the same sequence. The study uses a five-level model built based on Van Hiele's model in Table 1.

Table 1: Van Hiele's model in coordinate geometry

Levels	Description
Level 1: Visualization	Students observe the visual representations of objects, samples in numerical form. Students can identify, compare and group, and manipulate individual objects based on visualization.
Level 2: Analysis	Students can identify and describe objects through their characteristics and start analyzing objects. Students can connect characteristic properties to form concepts.
Level 3: Informal deduction	Students can receive and understand exact definitions. Students can learn more complex properties, classify and perform calculations. Students can perform thinking "if ... then ...", logical reasoning and reasoning about the nature and relationship of objects. General principles appear when calculating.
Level 4: Formal deduction	Students can make claims and perform proofs to determine the correctness of assertions.
Level 5: Rigor	Students can learn geometry without reference models and have the ability to reason through applying definitions, axioms and theorems.

2.3 The learning stages according to Van Hiele's model

Based on the impact of teaching on students' learning, Van Hiele's theory emphasizes that the teaching process should be organized into five learning stages to achieve each level of thinking. According to Van Hiele (1986), students have to go through many learning stages to reach their thinking levels in a specific learning topic. That is, to restart the learning stages when students do not understand a certain problem. These stages are described in Table 2 based on the synthesis from the studies of Hoffer (1983), Van Hiele (1986) and Teppo (1991).

Table 2: Learning stages according to Van Hiele's model

Learning stages	Description
Information	Students get acquainted with the learning content.
Guided orientation	Students are familiar with the ideas.
Explication	Students are aware of the initial relationships and begin to analyze based on existing knowledge.
Free orientation	Students can choose activities and self-orient the related framework.
Integration	Students can summarize, integrate, reflect, describe and apply the knowledge they have learned.

Regarding combining five levels of geometric thinking of Van Hiele's model and the five learning stages above, Table 3 presents descriptions of the activities of teachers and students in each learning stage corresponding to each level of thinking.

Table 3: Integration of learning stages and Van Hiele's model

Stages	Activities of the teacher	Activities of students
Information Level 1: Visualization	The teacher asks questions to help students recall relevant information associated with reality. The teacher presents new objects, creating conditions for students to observe, identify and classify visually. The teacher asks open-ended questions to identify students' orientations and concepts.	Students recall relevant information in practice. Through discussion, students discover the relationship between the object with their visual knowledge and ask exploring questions. Students visualize the objects according to what they observe. Students get acquainted with the object and start to explore its structure.
Guided orientation Level 2: Analysis	Teachers organize guided activities to help students become familiar with the properties of new concepts they have just discovered in visual form. The teacher can guide students to conduct a preliminary classification of objects.	Students perform tasks to help them discover the hidden relationships of the object. Students make a preliminary outline of the relationships between objects, properties of groups of objects.
Explication Level 3: Informal deduction	The teacher emphasizes vocabulary after students are familiar with the concept. Results should be as explicit as possible.	Students' experience is linked with symbols and language.
Free orientation Level 4: Formal deduction	Teachers can propose other problems for which students have not yet learned the static solution method.	Students perform more complex tasks, helping them master the network of relationships in the content to be learned. Students understand the properties learned but need to develop fluency in understanding relationships in different situations.
Integration Level 5: Rigor	The teacher provides students with a summary of what they have learned so far in class. During this period, the teacher does not give any new concepts, but, instead, summarizes what students have learned.	Students summarize what they have learned and enter information into the brain. Students make conclusions and consolidate or adjust math solutions.

(Source: Fuys et al., 1988; Masilo, 2018).

The purpose of the study and the research questions

According to this fact, straight-line equations are beneficial for forming and promoting students' mathematical reasoning and thinking skills. This study aims to form and improve these skills in 10th-grade students by teaching topics

involving straight-line equations. In order to achieve this goal, it is necessary to research in order to answer the following questions:
1. What topics in straight-line equations do students learn about in the 10th-grade math textbook?
2. Is there an effective way for students to make progress in their mathematical knowledge if they learn about straight-line equations during the learning stages of Van Hiele's model?
3. What strategies did students use to improve their mathematical thinking and reasoning abilities after learning from the stages above? Is it tough for them to think and reason mathematically?

3. Methods

Some of the contents of this section are presented in chronological order, including research samples, instruments, data collection and analyzing methods, and other related information. A study design that included a pre-test, an intervention, and a post-test was implemented to accomplish the research mentioned above's goal. Such research designs, which allow for assessing the effectiveness of educational innovations, are extremely common in educational research (Dugard & Todman, 1995; Tesch, 2016). A design similar to this one was developed by Papadakis et al. (2016) to develop mathematical competency in kindergarten through a realistic mathematics education approach.

3.1 Participants

The experiment was conducted from January 11, 2021, to January 27, 2021, at Doan Van To High School, Cu Lao Dung District, Soc Trang City, Vietnam. Experimental subjects include 44 students of 10A2 (experimental group) and 40 students of 10A3 (control group). The choice of students to participate in research was due to their interest and willingness to participate in active teaching activities.

3.2 Instruments

In the previous experiments, the students who participated took the post-test for the two groups. The preliminary research was conducted to see if the hypothesis was correct following the post-test and pre-test exams. To test if the experiment would be successful, validation and testing were required first. Researchers undertook the effort by developing proper and high-quality instruments. Two experts in the mathematics education field felt that the tests were reliable. The instruments and research were evaluated, and various changes were implemented. Experts verified that the instrument had not been revised further, and each one stated that the instrument was appropriate. They ultimately agreed to review the tests as they saw the relevance to the research topic. Furthermore, researchers can assess the coverage of academic content and skills as well. In this study, a more accurate response was made to measure a student's ability to think and to explain in different mathematical formats, including the use of visualization, analysis, informal deduction, formal deduction and the rigor of the conceptual presentation, in order to solve a specific mathematical problem linked to straight-line equations.

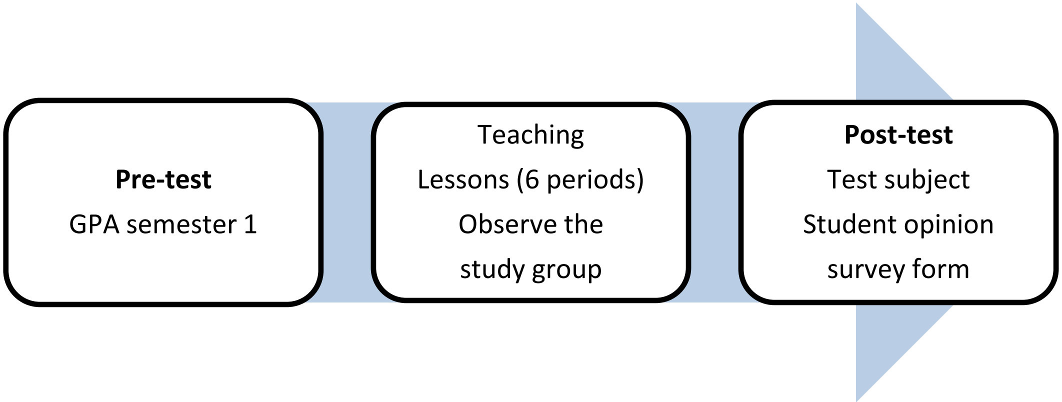


Figure 1: Experimental process and instruments

For the post-test, the following were rankings of scores used in evaluating the test results in the knowledge in Table 4 and Table 5.

Table 4: Ranking of post-test results according to each level of thinking and reasoning

Level	Score sum	Poor	Medium	Very good	Excellent
Level 1: *Visualization* Questions 1 and 2	10	0	1-4	5-8	9-10
Level 2: *Analysis* Questions 3 and 4	20	0	1-8	9-16	17-20
Level 3: *Informal deduction* Questions 4 and 5	45	0	1-16	17-32	33-45
Level 4: *Formal deduction* Question 6	15	0	1-6	7-10	11-15
Level 5: *Rigor* Question 7	10	0	1-5	6-10	---

Table 5: Ranking of post-test results

Score sum	Rating
0-34 points	Poor
35-64 points	Medium
65-79 points	Very good
80-100 points	Excellent

Regarding assessing the attitude, the student survey form was designed with ten questions on the Likert scale with five levels: Totally disagree - Disagree – Neutral - Agree - Totally agree.

3.3 Data collection and analysis

Table 6: Collection of experimental data

Contents	Experimental Group	Control Group
Average scores of semester 1	x	x
Post-test results after the experiment	x	x

Observe the study group	x	x
Results of students' opinion survey	x	-

From Table 6, data were collected based on average score results of semester 1 (replacing pre-test), results of post-test (post-test), study group observations and survey results of students. Data were tackled by quantitative analysis (t-test using SPSS 20 software) and qualitative analysis.

4. Results and discussion

Before giving detailed results, some preliminary results from classroom observations were noted. The lessons in the experimental and control groups were analyzed and compared to identify the best teaching methods, learning methods, acquired skills, learning content, and group atmosphere. Experimental teaching methods positively impacted students' learned contents, acquired skills, and learning attitudes. The process was created to integrate with Van Hiele's levels to help students go from the most basic levels of thinking and reasoning to the most complex. Thus, the group atmosphere and the learning attitude of the experimental group students were more comfortable and positive.

4.1 Pre-test results

The study used the average score of mathematics in the first semester of students to verify the qualifications of the experimental group and the control group. The independent t-test method was used to test the hypothesis that the average difference in mathematics of the experimental and control groups was not significantly different. Table 7 and Table 8 display the average mathematics descriptive and t-test results of experimental and control groups.

Table 7: Descriptive statistics of pre-test results

	Number N	Mean	Std Dev	Std Err	Minimum	Maximum
Experimental Group	44	6.4159	1.14444	0.17253	3.5	9.2
Control Group	40	6.1525	1.11055	0.17559	3.8	8.5

Table 8: Independent t-test results

Variances	t Stat	p-Value – 2 tailed (Sig.)	Mean Difference
Equal	1.068	0.288	0.26341

An independent t-test was used to test the significance of the average value difference between the experimental and control groups with equal variance hypotheses. Accordingly, with the significance level $\alpha = 0.05$, the critical value p = 0.288 was greater than 0.05. As a result, the average score values between the experimental and control groups were not significant. In other words, the pre-test results were speculated that the qualifications of the experimental group and the control group were equivalent.

4.2 Post-test results

The independent t-test method hypothesized that the average score of the post-experimental test of the experimental class was higher than that of the control class. Table 10 and Table 11 show descriptive statistics and t-test after the experimental and control groups.

Table 10: Descriptive statistics of post-test results

	N	Mean	Std Dev	StdErr	Minimum	Maximum
Experimental group	44	69.2727	11.70443	1.76451	37	87
Control group	40	39.8750	9.42497	1.49022	24	65

Table 11: Independent t-test results

Variance	t Stat	p-Value – 2 tailed (Sig.)	Mean Difference
Equal	12.598	0.000	29.39773

The value of influence level (SMD), according to Cohen, was 3.12 > 1.00. Therefore, it could be concluded that the influence of observed effects was very large. On the other hand, the independent t-test was used to verify the average score difference between the experimental and control groups with the null hypothesis. Correspondingly, with the significance level, $\alpha = 0.05$ and the critical value $p = 0.000$ was less than 0.05. Hence, the difference in average score values between the experimental group and control group was significant. In other words, the post-test results of the experimental group were significantly higher than the control group. The following is a schematic showing the score differentiation of the experimental and the control groups.

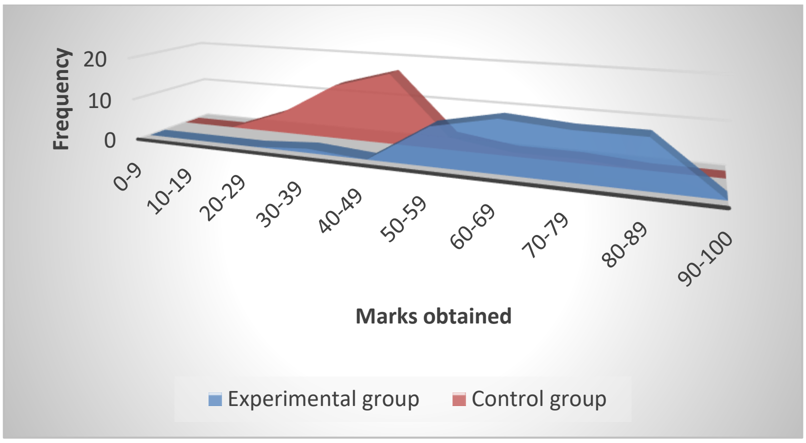


Figure 2: Diagram of score differentiation after the experiment

Figure 2 expresses that there was a clear difference in score differentiation between the experimental and control groups. Specifically, the experimental group had uniform point differentiation, concentrating at relatively high points (over 50 points). Meanwhile, the scores in the control group were distributed at many different levels of high and low, and there was a significant difference between the levels, especially the number of students reaching high scores (over 50 points) was relatively high. The checked scores of experimental and control groups were classified as shown in Table 12.

Table 12: Results of post-test rating

Rating	Poor	Medium	Very Good	Excellent
	0-34 points	35-64 points	65-79 points	80-100 points
Experimental Group	0	14	19	11
	0.00%	31.82%	43.18%	25.00%
Control Group	13	25	2	0
	32.50%	62.50%	5.00%	0.00%

Table 12 indicates that the percentage of assignments rated Poor in the experimental group was much lower than the control group. In the control group, the percentage of worksheets with the Medium rating was 62.5%, almost two times higher than that of the experimental group and accounted for most of the total number of tests. For the Very Good category, in the experimental group, 43.18% of the exercises with the Very Good category were significantly higher than the control group. At the same time, 25% of the worksheets achieved Excellent in the experimental group, and none of the papers achieved Excellent in the control group. Thus, none of the experimental worksheets were rated Poor, and the majority of the worksheets achieved either Very Good or Excellent (accounting for 68.18%), while, in the control group, most of the worksheets were rated Poor and Medium (95%). There are grounds to say that the learning results of the linear equation in the experimental class were significantly better than the control class. The post-test was designed with many different questions to evaluate and classify students' mathematical thinking and reasoning levels. The following were students' problems from experimental and control groups based on Van Hiele's model.

4.2.1 Level 1: Visualization

The level of visualization was assessed through Question 1 and Question 2. Question 1 asked students to apply practical experience and understanding of the features of the direction vector, normal vector, and the coefficient of the internal angle relationship with a straight line. Moreover, Question 2 was designed to evaluate the ability to identify and classify the straight-line equation types based on the algebraic features of the equation.

Table 13: Results of Level 1 - Visualization

Rating	Poor	Medium	Very Good	Excellent
	0 points	**1-4 points**	**5-8 points**	**9-10 points**
Experimental Group	0	0	2	42
	0.00%	0.00%	4.55%	95.45%
Control Group	0	2	14	24
	0.00%	5.00%	35.00%	60.00%

Table 13 reveals that most of the experimental group students met the visualization level requirements well. Specifically, 95.45% of the worksheets got Excellent, and the rest of them got Very Good. Meanwhile, 60% of the controls in the control group achieved Excellent, and 5% of the assignments were Medium. According to the above findings, it is clear that students in the experimental group had the same amount and level of thinking ability at the visual level as students in the control group. In this environment, this provided the foundation for a higher-order capacity of student thinking and reasoning. Many students still did not fully understand the practical visual aspects of the direction vectors, orthogonal vectors and angular coefficients, and the algebraic characteristics of straight-line equations for the control group. This level of differentiation could lead to substantial inequality by the next level.

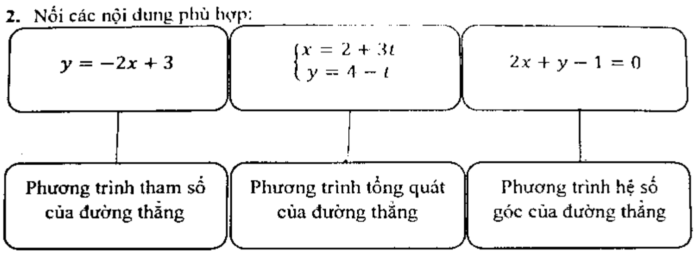


Figure 3: Student DC40's lesson - The answer was not good

Figure 3 expresses a faulty assignment that many of the control group's assignments had. Accordingly, many students had confusion between direction vectors and perpendicular vectors of the line. Figure 4 observes that the level of identification of students for types of linear equations was still limited. The fact that students failed to recognize and differentiate between types of straight-line equations may harm their ability to identify, write, and use the form of straight-line equations in different subject areas.

4.2.2 Level 2: Analysis

The student's level of analysis was assessed through Question 3 and Question 4. Question 3 was designed to test students' understanding of the direction vector and normal line vector characteristics, requiring them to identify and classify the given vectors based on their features and explain the classification basis. In Question 4, requirements were given to evaluate two levels of analysis and informal deduction.

Table 14: Results of Level 2 -Analysis

Rating	Poor	Medium	Very Good	Excellent
	0 points	1-8 points	9-16 points	17-20 points
Experimental Group	0	0	4	40
	0.00%	0.00%	9.09%	90.91%
Control Group	0	0	31	9
	0.00%	0.00%	77.50%	22.50%

Table 14 displays that the results of students' tests in both groups for the level of analysis achieved Very Good and Excellent. The percentage of assignments with excellent scores accounted for the majority (90.91%) in the experimental group. Meanwhile, most of the tests in the control group achieved Very Good (77.50%). Thus, most experimental group students met the analysis level requirements, such as identifying, classifying, pointing out the characteristics, and analyzing some basic elements of the subjects. For the students to understand how to think and reason, this helped them expand critical thinking and cognitive skills.

Accordingly, student TN40 in Question 1 had confusion between the actual image of the direction vector and the perpendicular vector of the line, leading to the correct classification of the vector group in Question 3. Meanwhile, student DC08's exercise was a typical example of many exercises in the control group. The solution of DC08 in Figure 4 reveals that students did not master the characteristics of the directional vectors and perpendicular vectors of the line, so the students gave the excess or lack of common points needed to group the vectors. For the analytical content in Question 4, most students in both groups could identify the initial elements of linear equations. Nonetheless, many students in the control group had difficulty in simply converting the straight-line elements. Through observing students' work in the test, it could be seen that many students were confused in determining which initial factors could be pointed out directly from the equation of the straight line.

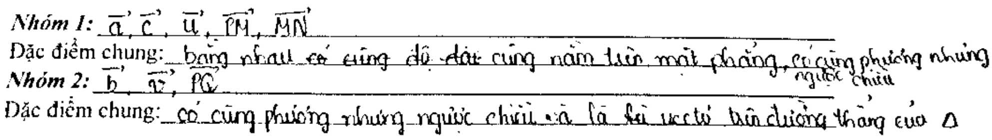


Figure 4: StudentDC08's homework - Not good answer

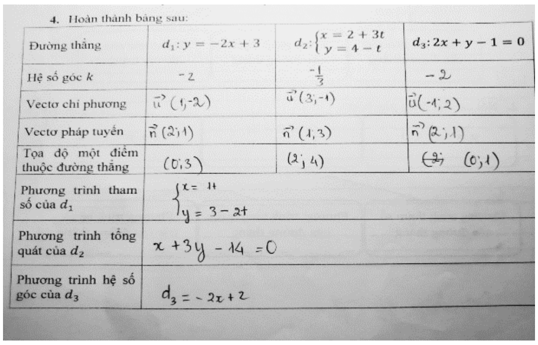
4.2.3 Level 3: Informal deduction

Regarding evaluating students' ability to think and reason mathematically at the level of informal deduction, Question 4 and Question 5 were designed to require students to perform calculations and transformations between types of linear equations, choose and apply the appropriate rules to determine the relative position between two lines, and simple reasoning to determine the relative position between the lines with the similar elements.

Table 15: Results of Level 3 – Informal deduction

Rating	Poor	Medium	Very Good	Excellent
	0 points	1-16 points	17-32 points	33-45 points
Experimental Group	0	4	23	17
	0.00%	9.09%	52.27%	38.64%
Control Group	0	22	18	0
	0.00%	55.00%	45.00%	0.00%

Data from Table 15 indicate that most of the worksheets in the experimental group scored Very Good and Excellent for the informal deduction level, with only 9.09% of the total rated Medium. Meanwhile, the control group mainly ranked Medium (55%) and Very Good (45%). To be classified as Very Good and Excellent at this level, students had to meet quite well the requests of skills and knowledge mentioned above; the answer by TN24 (Figure 5) was a typical lesson job. The poor classification results could be that the students did not meet the analysis requirements for the control group.



4. Hoàn thành bảng sau:

Đường thẳng	$d_1: y = -2x + 3$	$d_2: \begin{cases} x = 2 + 3t \\ y = 4 - t \end{cases}$	$d_3: 2x + y - 1 = 0$
Hệ số góc k	-2	$-\frac{1}{3}$	-2
Vectơ chỉ phương	$\vec{u}(1;-2)$	$\vec{u}(3;-1)$	$\vec{u}(-1;2)$
Vectơ pháp tuyến	$\vec{n}(2;1)$	$\vec{n}(1;3)$	$\vec{n}(2;1)$
Tọa độ một điểm thuộc đường thẳng	$(0;3)$	$(2;4)$	$(0;1)$
Phương trình tham số của d_1	$\begin{cases} x = t \\ y = 3 - 2t \end{cases}$		
Phương trình tổng quát của d_2	$x + 3y - 14 = 0$		
Phương trình hệ số góc của d_3	$d_3 = -2x + 2$		

Figure 5: Student TN24's good answer

Through the summary of the worksheets in both groups for Question 5, it could be noticed that most students chose to address the equation system containing two general equations of the line and based on the solutions of the system of equations to infer the relative position of two lines. In particular, some students in the experimental group displayed their creativity in thinking when solving problems. Although students still made some mistakes in mathematical representations, the idea of solving confirmed that students understood the relationship between the line and the direction vector and recognized the system of equations based on the coefficient. In the content that required students to argue to make conclusions about the relative position of two lines (question c), many students in the experimental group could make inferences based on parallel and intersect factors of the lines and argue closely to conclude. Meanwhile, control group problems could not help people determine a solution or provide coordinates between two lines. Thus, compared with the control group results, there are grounds to believe that the pedagogical methods applied in teaching in the experimental group impacted the growth of thinking and mathematical reasoning; these students studied at the level of informal deduction.

4.2.4 Level 4: Formal deduction

Question 6 was designed to evaluate students' ability to think and reason at the level of formal deduction; this question required students to perform analysis, synthesis, reasoning and reasoning to justify a claim. Due to the hierarchy between the levels according to Van Hiele's model, the formal deduction was a level requiring high order thinking to which not all students could respond well.

Table 16: Results of Level 4 - Formal deduction

Rating	Poor	Medium	Very Good	Excellent
	0 points	**1-6 points**	**7-10 points**	**11-15 points**
Experimental Group	4	23	17	0
	9.09%	52.27%	38.64%	0.00%
Control Group	39	1	0	0
	97.50%	2.50%	0.00%	0.00%

Table 16 expresses that the students in the experimental group achieved significantly better results than the control group students at the level of formal deduction. Hence, most of the worksheets in the experimental group were rated Average (52.27%) or Very Good (38.64%), and only 9.09% were rated Poor.

Accordingly, the exercises with Medium or above all determined the direction of solving the problem well and choosing the correct reasoning (using normal-line vectors of the straight line). However, they did not give complete reasoning with valid reasoning, and only error in mathematical representation was evaluated as a Very Good test (see Figure 6). Meanwhile, none of the control papers were rated Very Good, and 97.50% of the worksheets in the control class were rated Poor; most of the students did not give solutions, and some students gave the wrong argument. It is worth noting that none of the worksheets in both groups achieved Excellent, which meant that the students in the experimental group did not fully meet the requirements to be satisfied while arguing. It is possible to explain the reason based on the limited lesson time and that the frequency of students solving similar problems was quite small. Thus, it could be concluded that teaching designs for students in the experimental group achieved positive effects in developing students' ability to think and reason.

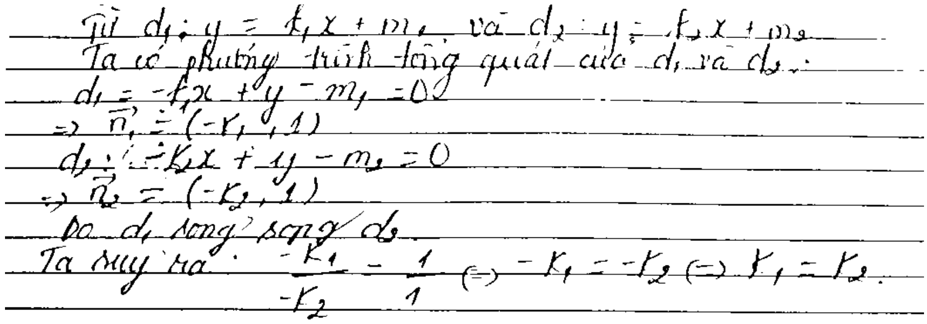


Figure 6: Student TN42's good answer

4.2.5 Level 5: Rigor

According to the Van Hiele model, students must have a good knowledge system for accurate thinking and mathematical thinking and knowledge to explain and solve real problems for high school students. Question 7 was designed to evaluate the level of accuracy of students, and the results were ranked at three levels of Poor, Medium and Very Good.

Table 17: Results of Level 5 - Rigor

Rating	Poor	Medium	Very Good	Excellent
	0 points	**1-5 points**	**6-10 points**	**---**
Experimental Group	3	38	3	---
	6.82%	86.36%	6.82%	---
Control Group	36	4	0	---
	90.00%	10.00%	0.00%	---

Table 17 shows that the experimental group results on the rigor level were better than that of the control group. In the control group, 90% of the worksheets did not answer this question, and only 10% of the worksheets mentioned how to reduce the slope but did not link it to the lesson and had no students who met the requirements well. The percentage of those in the control group who did not provide a solution was far lower in the experimental group. Additionally, 86.36% of the experimental group's worksheets had come up with suitable plans but had not yet applied the learned knowledge to explain the reasons for choosing that plan. The assignments rated Medium in both groups were mainly given by students' practical experience, i.e., "if you want to reduce the slope, you will increase the length and decrease the height of the slope," but this was not the case, so the result was as expected in Question 7.

On the other hand, a remarkable result was that for three students in the experimental group (accounting for 6.82%), creating a relationship between knowledge of the slope and slope and real-world situations was given. This result revealed that the pedagogical effects of enhancing thinking and reasoning at the level of rigor had been effective for some students. The lesson of student TN43 was a test that satisfies the expectations of the question (see Figure 7). As a result, the slope equation for the tangent line's angle was formulated to lower the line's gradient. Meanwhile, the students' problem was an example for the exercises with good ideas but based on experience, and there was no explanation based on the knowledge of the coefficients.

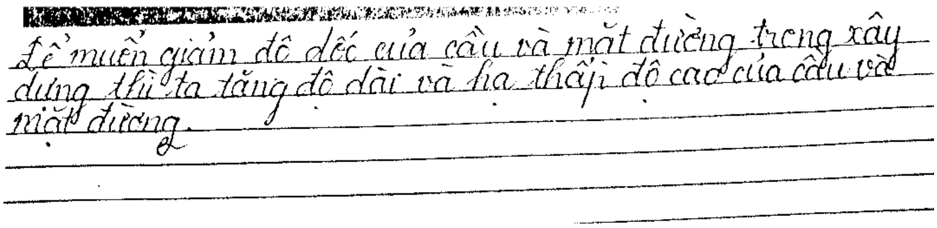


Figure 7: Student answer 14 - The answer with good ideas

Through analyzing the performance of two classes for each level according to Van Hiele's model, it could be concluded that students in the experimental class had significantly better performance in each level than the control class (See Figure 8).

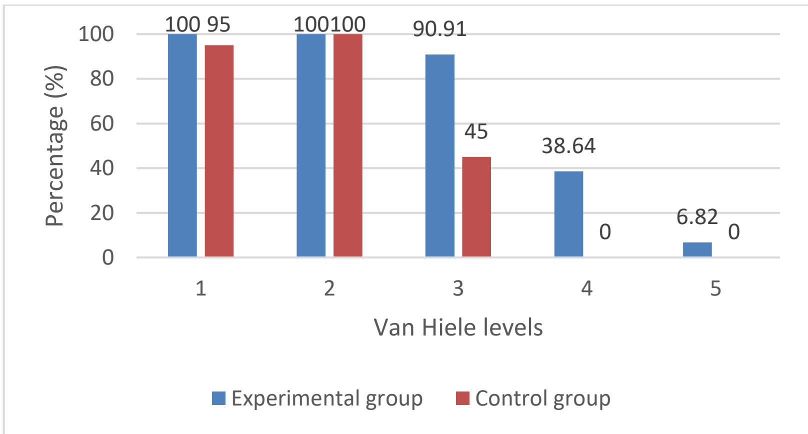


Figure 8: Graph of the rate of students performing very good and excellent in the levels of mathematical thinking and reasoning according to Van Hiele's model

As a result, the percentage of students responding well to the levels of visualization and analysis in both groups was quite equal and accounted for almost an absolute rate. However, according to the increasing informal to formal deduction and rigor levels, the disparity between the two groups increased. This result was probably due to the strict sequence and hierarchy of Van Hiele's mathematical thinking model. Accordingly, many students in the control group did not respond well to the level of analysis, leading to difficulties in higher levels for students. Alternatively, the higher the level of thinking required students to be trained under the oriented teaching organization of teachers. For this reason, the appropriate application of methods to improve mathematical thinking and reasoning according to Van Hiele's model could be a good orientation for teachers in teaching to enhance mathematical reasoning and thinking competencies for students.

4.3 Results of students' opinion survey

After completing the lesson of the straight-line equations, 45 students of the experimental group participated in giving comments on the lessons by answering the survey. The students' opinion survey included ten questions based on the Likert scale with five levels (Totally disagree - Disagree - Neutral - Agree - Totally agree) to collect students' opinions about the learning efficiency and the interest level of students in the lessons.

Question 1: I like the lessons in straight-line equations

Table 18: Survey results in Question 1

Levels	f	%
Totally disagree	0	0.0%
Disagree	0	0.0%
Neutral	1	2.2%
Agree	12	26.7%
Totally agree	32	71.1%

Table 18 indicates that most of the students in the experimental group liked practical lessons of the straight-line equation, and none of the students said they did not like these lessons. This result was consistent with the observed manifestations of the experimental group students' learning attitudes, as shown in Table 9.

Question 2: I find that the process of organizing activities in these lessons helps me to study more effectively

Table 19: Survey results in Question 2

Levels	f	%
Totally disagree	0	0.0%
Disagree	0	0.0%
Neutral	0	0.0%
Agree	14	31.1%
Totally agree	31	68.9%

Statistical results in Table 19 reveal that all students in the experimental group found that organizing activities in practical lessons helped them learn more effectively. Of which, 68.9% of students completely agreed with this statement. This result was a meaningful response to the research, contributing to the effectiveness and feasibility of the designed lesson plan.

Question 3: I find that visual activities (realistic images, drawings, sample expressions, mind maps) help me approach and visualize new concepts more easily

Table 20: Survey results in Question 3

Levels	f	%
Totally disagree	0	0.0%
Disagree	0	0.0%
Neutral	3	6.7%
Agree	7	15.6%
Totally agree	35	77.8%

From the results in Table 20, it could be seen that the majority of students (accounting for more than 90%) agreed or completely agreed that the visual activities (realistic images, drawings, sample expressions, mind maps) were effective in helping children access and visualize new concepts more easily. These activities contributed to enhancing students' ability to express their thoughts and. At the same time, it supported students in achieving the first level of thinking of Van Hiele's model - an important foundation for higher thinking levels.

Question 4: I find that analytical activities (showing characteristics, properties and classifications) help me understand concepts and their relationships better

Table 21: Survey results in Question 4

Levels	f	%
Totally disagree	0	0.0%
Disagree	0	0.0%
Neutral	1	2.2%
Agree	14	31.1%
Totally agree	30	66.7%

Question 4 was given to survey students' opinions about the effectiveness of analytical activities. According to Table 21, 97.8% of students said that these activities supported them in better understanding concepts and relationships between concepts, and no students gave the opposite feedback. These activities also assisted students in learning how to arrive at the mathematical thinking and reasoning levels. This result corroborated the experimental tests of students in the experimental group on the level of analysis (see Table 14).

Question 5: I find that presenting definitions, formulas, and problem-solving methods (verbal, symbols and diagrams) helps me better generalize, synthesize and memorize new knowledge

Table 22: Survey results in Question 5

Levels	f	%
Totally disagree	0	0.0%
Disagree	0	0.0%
Neutral	0	0,0%
Agree	14	31.1%
Totally agree	31	68.9%

From the findings from Table 22, it is clear that 100% of students could generalize, synthesize and memorize new knowledge better thanks to presenting definitions, formulas, and mathematical solution methods such as verbal symbols and diagrams. By observing study groups, most students were interested in generalizing and synthesizing knowledge by symbols and diagrams.

Question 6: I find that solving and proofing activities help me practice my ability to analyze and synthesize related knowledge, analogical reasoning in problem-solving, reasoning and presenting steps

Table 23: Survey results in Question 6

Levels	f	%
Totally disagree	0	0.0%
Disagree	0	0.0%
Neutral	2	4.4%
Agree	13	28.9%
Totally agree	30	66.7%

Table 23 shows the student's opinions about the effectiveness of solving math problems and demonstrating in training their ability to analyze, synthesize knowledge, make an analogous inference, reason and present reasoning steps. Accordingly, 28.9% of students agreed, and 66.7% of students completely agreed with the effectiveness of these activities, and no student said they disagreed. These were activities under the level of informal deduction and formal deduction of Van Hiele's model. The statistical results in Tables 15 and 16 found that this response was appropriate. Overall, 90% of the experimental group students obtained Medium or Good for the formal deduction, but. in correlation with the control group, the positive impacts of these activities were noticeable.

Question 7: I find that reinforcement and realistic relationships help me codify the knowledge I have learned more effectively and better understand the relationship between the learned knowledge and real-world problems

Table 24: Survey results in Question 7

Levels	f	%
Totally disagree	0	0.0%
Disagree	0	0.0%
Neutral	1	2.2%
Agree	11	24.4%
Totally agree	33	73.3%

The activities of reinforcement and realistic relationships were an indispensable part of teaching, according to Van Hiele's model. It was found from Table 24 that 97.7% of students realized the effectiveness of this activity in helping students systemize the knowledge they had learned more effectively and be more aware of the relationship between the learned knowledge and practical problems.

Question 8: I find that participating in group activities and manipulating the flashcards stimulates learning excitement and helps me learn more actively

Table 25: Survey results in Question 8

Levels	f	%
Totally disagree	0	0.0%
Disagree	0	0.0%

Neutral	2	4.4%
Agree	11	24.4%
Totally agree	32	71.1%

According to Table 25, collaborative thinking and enhancing the thinking skills were effective ways to stimulate students' thinking process and learning excitement. More than 95% of the students found group activities with study cards highly effective, while no students had an opposite view.

Question 9: I find myself improving in math calculation, thinking and reasoning

Table 26: Survey results in Question 9

Levels	f	%
Totally disagree	0	0.0%
Disagree	0	0.0%
Neutral	2	4.4%
Agree	14	31.1%
Totally agree	29	64.4%

In order to guide students through the process of self-assessment, this question was presented. Accordingly, the data from Table 26 indicate that 95.5% of all students found themselves improving their math learning, especially in mathematical thinking and reasoning. Furthermore, no student found learning ineffective. The results were predicted to be as a result of the research.

Question 10: I want to learn similar lessons in other periods

Table 27: Survey results in Question 10

Levels	f	%
Totally disagree	0	0.0%
Disagree	0	0.0%
Neutral	1	2.2%
Agree	13	28.9%
Totally agree	31	68.9%

Question 10 shows the students' appreciation for practical lessons. That was why students wanted to learn similar lessons in other lessons. Table 27 shows that more than 97% of students agreed or strongly agreed to learn the same lessons, and no students disagreed. If the lesson structures were generalized, these lessons could have the same effect on other lessons.

5. Conclusion

With experimental results analyzed, it was concluded that the experimental class students performed better than the control class in displaying mathematical reasoning and thinking competencies. Most experimental worksheets produced acceptable or outstanding results, with no failures; this result was significantly higher than the control group, with more than 90% of the students have achieved poor and medium grades and did not have excellent work. The students achieved similar results at the analytical and visual levels; this gave them a solid platform to grow their analytical and visual thinking skills.

For the informal deduction level, the data revealed that the students in the control group did not meet the requests of this level well, while the students in the experimental group had differentiation. Overall, the student results were slightly above average, but a relatively small percentage of students scored below average. At this level, the results of the two groups had quite a large difference in the rate of homework with high results. For the formal deduction level, the students in the control group did not satisfy the degree demands, with most of the students ranked poorly and no students performing well or excellently. According to Van Hiele's model hierarchy, students would only progress in their thinking if they met previous learning objectives (Van Hiele, 1986).

Meanwhile, students in the experimental group could not excel, but the overall success rate was also rather low; this finding demonstrates that students had satisfied the mathematical thinking and reasoning requests in solving mathematical problems. At the level of rigor, the study gave three grades of rating, including poor, medium and very good. The statistical results reported that most students in the experimental group achieved the average, and a few students excelled in the need to make connections between knowledge learned and real-world problems. Although this result was not too high, the experimental group's significant improvement could correlate with the control group.

The analytical results revealed that, as levels increased, there was a growing disparity between the experimental and control groups' ratings. This effect was consistent with Van Hiele's model hierarchy and indicated the effectiveness of experimental teaching designs on developing students' mathematical reasoning and thinking abilities. This outcome is consistent with the findings from the studies by Gutierrez and Jaime (1998) and Salifu et al. (2018).

On the other hand, through observing the period and surveying students' opinions in the experimental group, it was observed that the organization of teaching according to the learning stages of Van Hiele's model and teaching methods had brought positive effectiveness. Consequently, the integrated teaching stages corresponding to the levels of mathematical thinking created conditions for students to sequentially perform the necessary processes to train their ability to think and reason mathematically. Additionally, the group learning atmosphere was organized, especially visualization, mathematical software, languages, and various geometric representations and students' positive thinking. Interview, collaboration, and group work also contributed to improving the students' skills to think and reason mathematically with the principle of resonance in thinking. Research by Gunhan (2014), Hudson et al. (2015), Decy et al. (2018), Cesaria and Herman (2019) and Kovacevic (2019) also had relevant results.

Thus, applying Van Hiele's model and learning phases according to this model in teaching contributes to enhancing teaching effectiveness and promoting mathematical reasoning and thinking competencies for students. According to Van Hiele's model, mathematical thinking and reasoning levels guide teachers well in the lesson design process. In terms of psychology, at that point, forming knowledge for students is consistent with their cognitive abilities. In this way, the material is more digestible for students. As a result, they developed a better and more thorough understanding of topics. An implication was observed that one of the reasons for the difficulty of high school students in connecting mathematics to solving problems was the lack of reasoning skills (Jailani et al., 2020).

Only 84 students in the 10th grade were observed because this was a case study. Both the ability and willingness of subjects to learn differed dramatically between genders, and by their preferred learning style, they could be divided into groups. The study was based on enhancing students' abilities to think and reason mathematically. Also, it should be noted that the study content was connected to straight-line equations covered in the 10th-grade mathematics textbook.

Further studies can examine mathematical thinking and reasoning competencies related to these two types of competencies due to the close relationship between mathematical reasoning and thinking skills with problem-solving and modeling capacities. Moreover, it is possible to expand the research on Van Hiele's model to improve students' mathematical reasoning and thinking abilities in other fields such as algebra or calculus.

6. References

Abdullah, A. H., & Zakaria, E. (2013). The effects of Van Hiele's phases of learning geometry on students' degree of acquisition of Van Hiele levels. *Procedia - Social and Behavioral Sciences*, *102*, 251-266. https://doi.org/10.1016/j.sbspro.2013.10.740

Alex, J., & Mammen, K. J. (2018). Students' understanding of geometry terminology through the lens of Van Hiele theory. *Pythagoras*, *39*(1), 1-8. https://doi.org/10.4102/pythagoras.v39i1.376

Bell, A. E., Costello, J., & Kuchemann, D. E. (1983). *A Review of research in mathematical Education. Part A. Research on Learning and Teaching*. Nfer - Nelson.

Cesaria, A., & Herman, T. (2019). Mathematical reasoning in geometry learning using Information and Communication Technology (ICT). *Journal of Physics: Conf. Series 1157*(4), 1-6. http://dx.doi.org/10.1088/1742-6596/1157/4/042101

Cresswell, C., & Speelman, C. P. (2020). Does mathematics training lead to better logical thinking and reasoning? A cross-sectional assessment from students to professors. *PLoS ONE, 15*(7), 1-21. https://dx.doi.org/10.1371/journal.pone.0236153

Crowley, M. L. (1987). The Van Hiele Model of development of geometric thought. In M. M. Lindquist, & P. Shulte (Eds.), *Learning and teaching geometry, K-12, 1987 Yearbook*, 1-16. Reston, VA: National Council of Teachers of Mathematics.

Dhayanti, D., Johar, R., & Zubainur, C. M. (2018). Improving students' critical and creative thinking through Realistic Mathematics Education using Geometer's Sketchpad. *Journal of Research and Advances in Mathematics Education, 3*(1), 25-35. http://journals.ums.ac.id/index.php/jramathedu.

Drijvers, P., Kodde-Buitenhuis, H., & Doorman, M. (2019). Assessing mathematical thinking as part of curriculum reform in the Netherlands. *Educational Studies in Mathematics, 102*, 435–456. https://doi.org/10.1007/s10649-019-09905-7

Drijvers, P. (2015). *Denken over wiskunde, onderwijs en ICT. [Thinking about Mathematics, education and ICT.]* Inaugural lecture. Utrecht, the Netherlands: Universiteit Utrecht. http://www.fisme.science.uu.nl/publicaties/literatuur/Oratie_Paul_Drijvers_facsimile_20150521.pdf.

Dugard, P., & Todman, J. (1995). Analysis of pre-test-post-test control group designs in educational research. *Educational Psychology*, *15*(2), 181-198. https://doi.org/10.1080/0144341950150207

Feza, N., & Webb, P. (2005). Assessment standards, Van Hiele levels, and grade seven learners' understandings of geometry. *Pythagoras*, *2005*(62), 36-47. https://doi.org/10.4102/pythagoras.v0i62.113

Fuys, D., Geddes, D., & Tischler, R. (1988). The Van Hiele model of thinking in geometry among adolescents. *Journal for Research in Mathematics Education. Monograph, 3*, 1-196. https://doi.org/10.2307/749957.

Gunhan, B. C. (2014). A case study on the investigation of reasoning skills in geometry. *South African Journal of Education,* *34*(2), 1-19. https://doi.org/10.15700/201412071156

Gutiérrez, A., & Jaime, A. (1998). On the assessment of the Van Hiele levels of reasoning. *Focus on Learning Problems in Mathematics Spring & Summer Edition, 20*(2 & 3), 27-46.

Hoffer, A. R. (1983). Van Hiele - based research in acquisition of mathematical concepts and processes. In R. Lesh & M. Landau (Eds.), *Acquisition of mathematics concepts and processes* (pp. 205-227). Academic Press Or1ando.

Hudson, B., Henderson, S., & Hudson, A. (2015). Developing mathematical thinking in the primary classroom: liberating students and teachers as learners of mathematics, *Journal of Curriculum Studies,* *47*(3), 374-398. https://doi.org/10.1080/00220272.2014.979233

Jailani, Retnawati, H., Apino, E., & Santoso, A. (2020). High school students' difficulties in making mathematical connections when solving problems. *International Journal of Learning, Teaching and Educational Research,* 19(8), 255-277. https://doi.org/10.26803/ijlter.19.8.14

Karadag, Z. (2009). *Analyzing Students' Mathematical Thinking In Technology-Supported Environments* [Doctoral dissertation, The University of Toronto]. https://tspace.library.utoronto.ca/bitstream/1807/19128/1/Karadag_Zekeriya_200911_PhD_thesis.pdf

Kovacevic, N. (2019). Spatial reasoning in mathematics. *Teaching And Learning Mathematics.* https://www.researchgate.net/publication/331430837_Spatial_reasoning_in_mathematics.

Land, J. E. (1990). *Appropriateness of the Van Hiele model for describing students' cognitive processes on algebra tasks as typified by college students' learning*. UMI Dissertation Services. Michigan.

Masilo, M. M. (2018). *Implementing Inquiry-Based Learning To Enhance Grade 11 Students' Problem-Solving Skills In Euclidean Geometry* [Doctoral dissertation, The University Of South Africa]. http://hdl.handle.net/10500/24966

Mason, J., Burton, L., & Stacey, K. (2010). *Thinking mathematically* (2nd ed.). Prentice Hall.

Ministry of Education and Training. (2018). *Mathematics General Education Curriculum*. Hanoi, Vietnam.

Mukono, S. (2015). *Grade 11 Mathematics Learners' Concept Images And Mathematical Reasoning On Transformations Of Functions* [Doctoral dissertation, The University Of South Africa]. https://core.ac.uk/download/pdf/43177597.pdf

Mumu, J., & Tanujaya, B. (2019). Measure reasoning skill of mathematics students. *International Journal of Higher Education,* *8*(6), 85-91. https://doi.org/10.5430/ijhe.v8n6p85

Niss, M., & Højgaard, T. (2011). *Competencies and Mathematical Learning Ideas and inspiration for the development of mathematics teaching and learning in Denmark*. English edition. Denmark: IMFUFA, Roskilde University.

Organization for Economic Cooperation and Development. (2018). *PISA 2021 Mathematics Framework (Draft)*. OECD Publishing. EDU/PISA/GB(2018)19.

Papadakis, St., Kalogiannakis, M., & Zaranis, N. (2016). Improving mathematics teaching in kindergarten with realistic mathematical education. *Early Childhood Education Journal*, *45*(3), 369–378.

Paul, R., & Elder, L. (2008). Critical thinking: strategies for improving student learning. *Journal of Developmental Education, 32*(2), 34-35.

Polya, G. (1963). On learning, teaching, and learning teaching. *The American Mathematical Monthly, 70*(6), 605–619. https://doi.org/10.1080/00029890.1963.11992076

Salifu, A. S., Yakubu, A. R., & Ibrahim, F. I. (2018). Van Hiele Geometric Thinking Levels of Pre-Service Teachers of E. P. College of Education, Bimbilla-Ghana. *Journal of Education and Practice, 9*(23), 108-119.

Sinwell, B. J. (2017). *Formative Assessment Strategies for Mathematical Thinking: A Qualitative Action Research Study* [Doctoral dissertation, University of South Carolina]. https://scholarcommons.sc.edu/etd/4112

Soares, L. (2012). *A 'Disruptive' Look at Competency-Based Education How the Innovative Use of Technology Will Transform the College Experience*. Center for American Progress.

Tall, D. (2002). The Psychology of Advanced Mathematical Thinking. In D. Tall (Ed.), *Advanced Mathematical Thinking* (pp. 3-24). Kluwer Academic Publishers.

Teppo, A. (1991). The Van Hiele levels of geometric thought revisited. *The Mathematics teacher, 84*(3), 210-221. https://doi.org/10.5951/mt.84.3.0210

Tesch, A. (2016). Implementing pre-post test designs in higher education evaluations. *New Directions for Evaluation*, *2016*(151), 85-96. https://doi.org/10.1002/ev.20195

Van Hiele, P. M. (1986). *Structure and Insight. A theory of Mathematics Education*. Academic Press Inc.

Vojkuvkova, I. (2012). The Van Hiele Model of Geometric Thinking *WDS'12 Proceedings of Contributed Papers, Part I*, 72–75.

Appendix 1 (Post-test questions)

ASSESSMENT TASK

STRAIGHT-LINE EQUATIONS

VAN TO HIGH SCHOOL

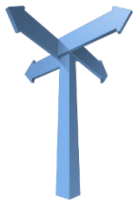
CLASS:_______

DATE:___/___/2021

QUESTIONS

(Students' part)

1. Match each of the following figures with an appropriate box: *(5 points)*



Direction vector of a line

Normal vector of a

Slope of a line

2. Match each of the following boxes with an appropriate box below: *(5 points)*

$y = -2x + 3$

$\begin{cases} x = 2 + 3t \\ y = 4 - t \end{cases}$

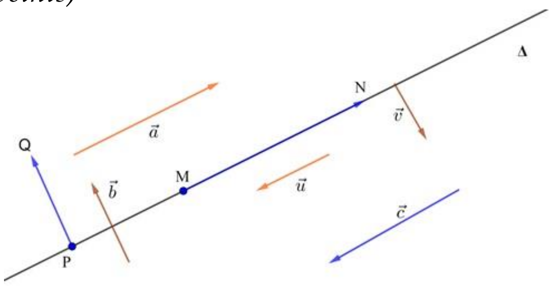
$2x + y - 1 = 0$

Parametric equation of a line

General equation of a

Slope-intercept equation of a line

3. Group the following vectors and define the similar characteristics of the vectors in each group. *(10 points)*



Group 1:______________________________________ *(2 points)*

Characteristics:______________________________________ *(3 points)*

Group 2:______________________________________ *(2 points)*

Characteristics:______________________________________ *(3 points)*

4. Complete the following table: *(35 points)*

Line	$d_1: y = -2x + 3$	$d_2: \begin{cases} x = 2 + 3t \\ y = 4 - t \end{cases}$	$d_3: 2x + y - 1 = 0$
Slope k	*(1 point)*	*(2 points)*	*(2 points)*
Direction vector	*(2 points)*	*(1 point)*	*(2 points)*
Normal vector	*(2 points)*	*(2 points)*	*(1 point)*
Coordinates of a point on the line	*(2 points)*	*(1 point)*	*(2 points)*
Parametric equation of d_1	*(5 points)*		
General equation of d_2	*(5 points)*		
Slope-intercept equation of d_3	*(5 points)*		

5. From the results of task 4 determine the relative positions between the following lines: *(20 points)*

a) d_1 and d_2: *(5 points) (Strategy: 2 points, Calculation: 2 points, Conclusion: 1 point)*

____

____

b) d_1 and d_3: *(5 points) (Strategy: 2 points, Calculation: 2 points, Conclusion: 1 point)*

____

____

c) Use the previous results to infer a conclusion about the relative position between d_2 and d_3: *(10 points) (Reasoning: 10 points; In case student solves by calculation: 3 points)*

____

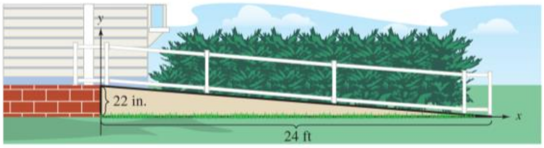
____

6. Given $d_1: y = k_1x + m_1$ and $d_2: y = k_2x + m_2, k_1, k_2 \neq 0$. Prove that $k_1 = k_2$ if these lines are parallel. Is there any other proof? If yes, provide these proofs. *(15 points) (Proof: 10 points, Other proofs: 5 points)*

____

____

7. The slope of a line has a close relationship with the concept of **slope** in practice. Steep sections of roads or bridges often cause difficulties for roadsters. Thus, in construction, if we want to reduce the slope of a road or a bridge, what strategies can we use? Apply the knowledge of the slope of a line to answer the question.



(10 points) (Strategies: 5 points, Associating with the knowledge of the slope of a line: 5 points)

____

____

- The end -

RESULTS
(*Teacher's part*)

Question	Mark	Level
1	5	Level 1: Visualization
2	5	Level 1: Visualization
3	10	Level 2: Analysis
4	35	Level 2: Analysis (10 points) Level 3: Informal deduction (25 points)
5	20	Level 3: Informal deduction
6	15	Level 4: Formal deduction
7	10	Level 5: Rigor
Total: 7	**Total:** 100 points	**Total:** 5 levels

STUDENT'S CODE:______________________________
Question 1: __________________________________
Question 2: __________________________________
Question 3: __________________________________
Question 4: __________________________________
Question 5: __________________________________
Question 6: __________________________________
Question 7: __________________________________
Other results:

____

____