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The Mathematical Beauty

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Abstract. Mathematics is a science. However, Mathematics has exceptional features that other sciences can hardly attain; for instance the beauty in cognitive development, in Mathematics applied in other fields such as Physics, Computer Science, Music, Fine Art, Literature, etc… Mathematical beauty manifests itself in many forms and is divided into many different categories. Mathematical beauty can be divided into inner and outer beauty, or it can be categorized by fields or divided into the beauty in method, in problem development, and in mathematical formulas. The charactersitics of mathematical are repetition, harmony and Non-monotonicity. Beauty is a vague concept. It is not easy to define, measure, or estimate.

Keywords. Mathematical beauty, outer beauty, inner beauty, mathematical formula.

1. Introduction

Mathematical beauty is the notion that some mathematicians generally use to describe mathematical results, methods,… which are interesting, unique, and elegant. Mathematicians often regard these results and methods as elegant and creative. They are often likened to a good poem or a passionate song. Mathematical beauty manifests itself in a variety of ways. It might be cognitive, or it might be in the form of symmetrical shapes. It might be visible or hidden away. This is a broad notion that involves a large number of aspects of life, in science and in art.

2. Main results

2.1. The concept of beauty

It is quite difficult to define beauty. It is an aesthetic category. It affects the human senses and brings about feelings of joy and excitement, and creates perfection and meaningfulness.

Mohammed said: “If I had only two loaves of bread, I would barter one to nourish my soul.” (Huntley, 1970)

Richard Jefferies wrote: “The hours when we absorbed by beauty are the only hours when we really live … These are the only hours that absorb the soul and

fill it with beauty. This is real life, and all else is illusion, or mere endurance." (Huntley, 1970)
The *Shorter Oxford English Dictionary* states that, beauty is "That quality or combination of qualities which affords keen pleasure to the senses, especially that of sight, or which charms the intellectual or moral faculties." (William, 2002)
Aquinas said "Beauty is that which pleases in mere contemplation" (Viktor, 2012)
According to an English proverb, "Beauty is in the eyes of the beholder." Whether something is beautiful or not is dependent on a person's perception. One might regard a painting as pretty and meaningful, while another regards the same painting as ugly and meaningless. A beautiful painting or statue is not likely to be loved by all. On the other hand, when it has earned the love of all people, whether the painting is beautiful or not is of little importance. Beauty is a vague concept. It is not easy to define, measure, or estimate.

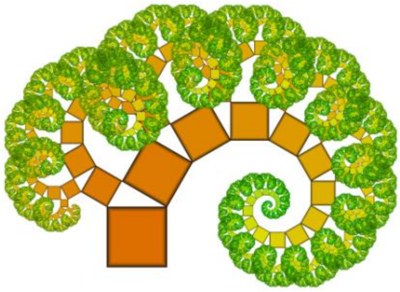
2.2. The concept of mathematical beauty

There are many different views on mathematical beauty. It appears in a variety of fields, from natural sciences to social sciences, and in everyday life. According to Bertrand Russell, mathematical beauty is defined as follows: "Mathematics, rightly viewed, possesses not only truth, but supreme beauty – a beauty cold and austere, like that of sculpture, without appeal to any part of our weaker nature, without the gorgeous trappings of painting or music, yet sublimely pure, and capable of a stern perfection such as only the greatest art can show. The true spirit of delight, the exaltation, the sense of being more than Man, which is the touchstone of the highest excellence, is to be found in mathematics as surely as poetry." (Russell, 1919)
Edna St. Vincent Millay said "Euclid alone has looked on beauty bare …" (Huntley, 1970)
Rota wrote: "We think to instances of mathematics beauty as if they had been perceived by an instantaneous realization, in moment of truth, like a light-bulb suddenly being it. All the effort that went in understanding the proof of a beautiful theorem, all the background material that is needed if the statement is to make any sense, all the difficulties we met in following an intricate sequence of logical inferences, all these features disappear once we become aware of the beauty of a mathematical theorem and what will remain in our memory of our process of learning is the image of an instant flash of insight, of a sudden light in the darkness" (Viktor, 2012)
From our point of view, the aesthetic element of mathematical beauty depends on our outlook on the perfection of methods, problems, as well as on the perspective of the mathematical subject. Mathematical beauty is the result of discovering both the inner and outer link between mathematical objects and phenomena.

2.3. The characteristics of mathematical beauty

2.3.1. Repetition

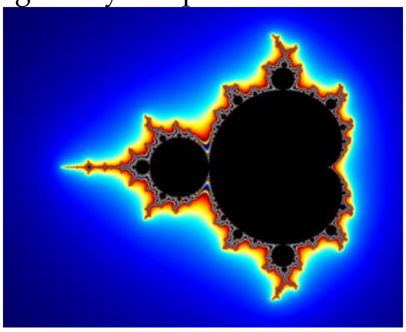
As stated above, "Beauty is in the eyes of the beholder", but the creator of a problem, a formula or a drawing can only be considered successful when his creations are acknowledged as being beautiful.
The first characteristic of mathematical beauty is repetition.


Picture 1. Pythagorean Tree

A piece of music has repetitive beats in addition to choruses. A poem has repetitive rhymes.

The most common and obvious feature of repetition is symmetry, which is when an object has similar parts that can rotate or swap places without changing the overall shape of the object itself.

There might be no other field in Mathematics that has as beautiful symmetrical shapes as Fractals. The Pythagorean Tree above, as well as the following Mandelbrot set, expresses the captivating beauty of repetition.


Picture 2. Mandelbrot set

2.3.2. Harmony

Harmony is an abstract concept. There is a combination of elements that gives off the impression of being beautiful. Any two things are considered harmonious when they are in tune with each other.

For example, if the movements of a swimmer (hands, legs, breathing, etc.) correspond, his posture will look graceful and elegant; on the other hand, if his movements are messy and out of tune, which indicates a lack of harmony, it is difficult to stay afloat. In a painting, if the most important visuals are shoved into one corner while the rest of the painting is blank, it is inharmonious, since the size of the piece is not proportionate to the content. In a piece of music, it is common that there are multiple notes sounding together at one time, rather than only one single note. If all those notes resonate (in a physical sense), they sound pleasant and harmonious, while separate notes not resonating make lousy sounds. A harmonious mathematical problem must have a graceful way of wording, creating a number of meaningful results. Take Fermat's Last Theorem as an example: *Prove that the Diophantine equation* $x^n + y^n = z^n$ *has no integer solutions for* $n > 2$ and $x, y, z \neq 0$. A problem is inharmonious when it has excessively complicated wording, and the solution uses too many unnecessary tricks.

2.3.3. Non-monotonicity

Amateur "artists" can imitate famous works of art; for example, the *Mona Lisa* by Leonardo de Vinci has been recreated numerous times by various artists. However, no matter how similar they are, the copies are always inferior to the original in some way. A great piece of art ought to have something new, different from its predecessors.

Even in the same piece of art, if a single motif, however interesting it might be, is repeated time and again, it can become monotonous. Therefore, it is necessary to change, to create an element of surprise, in order to generate interest among the audience. In Mathematics, applying a single method to a multitude of problems would be far more monotonous than using different methods for different problems.

2.3.4. Human-relatedness

It is easier for people to grasp things that can be linked to information already existing in their heads. Meanwhile, strange and random things that have no connection to anything cannot stir up emotions within a person. That is the reason why many paintings and sculptures have the human body as their main theme, since it is the most familiar thing to people. A painting or a sculpture of a "Martian", no matter how beautiful, could hardly garner interest, as a "Martian" is a foreign concept to humans.

Mathematical problems as well as topics have to be suitable for the person solving it. If he has the ability to understand the results, his interest will be piqued, and he will want to put more effort into his study. On the other hand, if he is unfamiliar with the knowledge, it is easier for him to give up. According to Vygotsky, a person who solves mathematical problems is only interested in the knowledge that is in his Zone of Proximal Development. Problems that are too familiar are simple and uninteresting, while ones that are too unfamiliar are too complex, and therefore also uninteresting.

2.4. Categorizing mathematical beauty

There are many ways to categorize mathematical beauty. It can be divided into inner and outer beauty, or it can be categorized by fields, such as mathematical beauty in Art, Computer Science, Physics or Music, etc. Or it can be divided into the beauty in method, in problem development, and in mathematical formulas.

2.4.1. Categorizing mathematical beauty according to method, problem development, and mathematical formulas

Mathematical beauty in method has the following characteristics:

- A proof that uses the additional assumptions or previous results.
- A proof that is quite simple.
- A new proof.
- A proof based on original insights.
- A proof can easily generalize to solve similar problems.
- A proof that might be long, but results in new, interesting and insightful results.

The following example illustrates the beauty in method. Our new proof for the Bouniakowsky inequality is as follows (published on Romanian Mathematical Magazine):

Problem 1 (*The CBS – inequality*)

Given $x_1, x_2, .., x_n; y_1, y_2, ..., y_n \in \square$. Prove that

$$\left(x_1^2 + x_2^2 + ... + x_n^2\right)\left(y_1^2 + y_2^2 + ... + y_n^2\right) \geq \left(x_1y_1 + x_2y_2 + ... + x_ny_n\right)^2.$$

The new solution is as follows

Case 1

If $x_1^2 + x_2^2 + ... + x_n^2 = 0$ or $y_1^2 + y_2^2 + ... + y_n^2 = 0$ we have Q. E. D.

Case 2

If $x_1^2 + x_2^2 + ... + x_n^2 \neq 0$ or $y_1^2 + y_2^2 + ... + y_n^2 \neq 0$ then we let

$$R_x^2 = x_1^2 + x_2^2 + ... + x_n^2; R_y^2 = y_1^2 + y_2^2 + ... + y_n^2 \text{ (1)}$$

We have

$$\begin{cases} x_1 = R_x \sin\alpha_1 \sin\alpha_2 ... \sin\alpha_{n-2} \sin\alpha_{n-1} \\ x_2 = R_x \sin\alpha_1 \sin\alpha_2 ... \sin\alpha_{n-2} \cos\alpha_{n-1} \\ x_3 = R_x \sin\alpha_1 \sin\alpha_2 ... \cos\alpha_{n-2} \\ ... \\ x_n = R_x \cos\alpha_1 \end{cases}$$

and

$$\begin{cases} y_1 = R_y \sin\beta_1 \sin\beta_2 ... \sin\beta_{n-2} \sin\beta_{n-1} \\ y_2 = R_y \sin\beta_1 \sin\beta_2 ... \sin\beta_{n-2} \cos\beta_{n-1} \\ y_3 = R_y \sin\beta_1 \sin\beta_2 ... \cos\beta_{n-2} \\ ... \\ y_n = R_y \cos\beta_1 \end{cases}.$$

We have

$$x_1y_1 = R_xR_y \prod_{k=1}^{n-2} \sin\alpha_k \sin\beta_k \sin\alpha_{n-1} \sin\beta_{n-1}; x_2y_2 = R_xR_y \prod_{k=1}^{n-2} \sin\alpha_k \sin\beta_k \cos\alpha_{n-1} \cos\beta_{n-1}.$$

Thus,

$$x_1y_1 + x_2y_2 \leq |x_1y_1 + x_2y_2| = \left|R_xR_y\right| \left|\prod_{k=1}^{n-2} \sin\alpha_k \sin\beta_k\right| . \left|\cos(\alpha_{n-1} - \beta_{n-1})\right|$$

$$\leq \left|R_xR_y\right| . \left|\prod_{k=1}^{n-2} \sin\alpha_k \sin\beta_k\right|.$$

From this relation, we have:

$$x_1y_1 + x_2y_2 + x_3y_3 \leq |x_1y_1 + x_2y_2 + ... + x_ny_n| \leq |R_xR_y| \text{ (2)}.$$

From (1) and (2), we have

$$\left(x_1^2 + x_2^2 + ... + x_n^2\right)\left(y_1^2 + y_2^2 + ... + y_n^2\right) \geq \left(x_1y_1 + x_2y_2 + ... + x_ny_n\right)^2. \text{ (}Q.\ E.\ D\text{)}$$

The equality happens if and only if $\frac{x_1}{y_1} = \frac{x_2}{y_2} = ... = \frac{x_n}{y_n}$.

The beauty in problem development is the beauty of creativity in Mathematics. Assimilating, specializing, and generalizing mathematical problems bring about a deep understanding about a subject and help a person to discover the hidden link between things. Through the results, the person will be able to realize the good and exciting things that are normally hard to see.

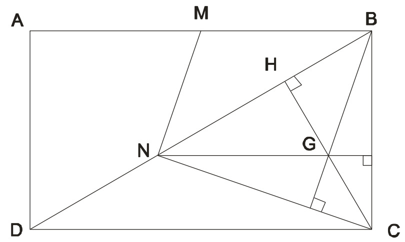
The following example demonstrates the beauty in mathematical problem development.

Problem 2

$ABCD$ is a rectangle. Let M be the midpoint of AB, let H be the foot of the perpendicular from C on BD, let N be the midpoint of DH. Prove that $CNM = 90^0$.

The following are some solutions

Solution 1 *(The synthetic method)*

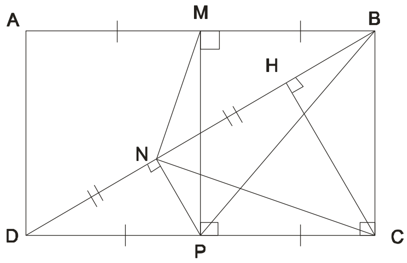


From N, draw $NG // DC$. By the midline theorem, we have:

$$NG // DC, NG = \frac{1}{2}DC.$$

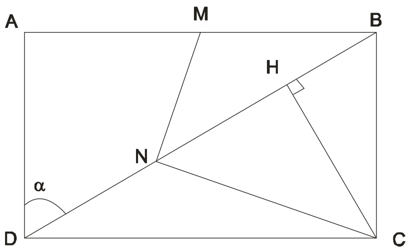
Thus $NG // MB$ and $NG = MB$ or $NGBM$ is a parallelogram. We have $MB \perp BC$, so $NG \perp BC$. Thus, G is the orthocentre of the triangle NBC. Thus, $BG \perp NC$. It follows $MN \perp NC$, i.e., $CNM = 90^0$.

Solution 2 *(The synthetic method)*



Let P the midpoint of CD. We have $PNB = PMB = PCB = 1v$. Thus, five points P, N, M, B, C lie on a circle with the diameter MC. Thus, we have $CNM = 90^0$.

Solution 3 *(The vectorial method)*



We have

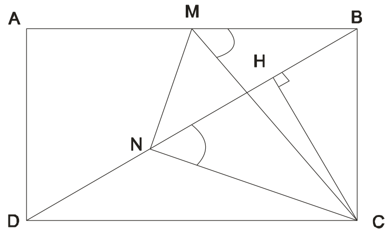
$$\overrightarrow{MN}\,.\,\overrightarrow{NC}=\frac{1}{2}(\overrightarrow{AD}+\overrightarrow{BH})\,.\,\frac{1}{2}(\overrightarrow{DC}+\overrightarrow{HC})=\frac{1}{4}(\overrightarrow{AD}+\overrightarrow{BH})\,.\,(\overrightarrow{HB}+\overrightarrow{BC}+\overrightarrow{DC})$$

$$=\frac{1}{4}\left(-AD\,.\,HB\,.\cos\alpha+AD^2+BH\,.\,BC\,.\cos\alpha-BH^2-BH\,.\,DC\,.\sin\alpha\right)$$

$$=\frac{1}{4}(CH^2-BH\,.\,DC\,.\,\frac{HD}{DC})=\frac{1}{4}(CH^2-BH\,.\,HD)=0.$$

Thus, $CNM=90^0$.

Solution 4 *(The trigonometric method)*



In order to prove $CNM=90^0$, we need to prove that $MBCN$ is a concyclic quadrilateral.

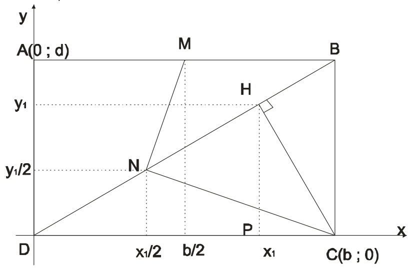
Indeed, we have

$$CAB=BDC\Rightarrow\frac{BC}{AB}=\frac{HC}{HD}\Rightarrow\frac{1}{2}\,.\,\frac{BC}{BM}=\frac{1}{2}\,.\,\frac{HC}{NH}$$

$$\Rightarrow\tan BMC=\tan BNC\Rightarrow BMC=BNC.$$

Thus, $MNCB$ is a concyclic quadrilateral, which is $CNM=90^0$.

Solution 5 *(The coordinate method)*



Consider the system of Cartesian coordinates Dxy as the above figure. We have

$$D(0\,;0),\,C(b\,;0),\,A(0\,;d),\,M(\frac{b}{2}\,;d),\,H(x_1\,;y_1),\,N\left(\frac{x_1}{2}\,;\frac{y_1}{2}\right).$$

The equation of the line MN is

$$\frac{x-\frac{x_1}{2}}{y-\frac{y_1}{2}}=\frac{\frac{x_1}{2}-\frac{b}{2}}{\frac{y_1}{2}-d}\Leftrightarrow x-\frac{x_1}{2}=\frac{x_1-b}{y_1-2d}y-\frac{y_1}{2}\,.\,\frac{x_1-b}{y_1-2d}$$

$$\Rightarrow y=\frac{y_1-2d}{x_1-b}x+\frac{y_1}{2}-\frac{x_1}{2}\,.\,\frac{y_1-2d}{x_1-b}.$$

The equation of the line NC is

$$\frac{x-b}{y}=\frac{x_1-2b}{y_1}\Rightarrow y=\frac{y_1}{x_1-2b}x-b\,.\,\frac{y_1}{x_1-2b}.$$

The necessary and sufficient condition for $MN \perp NC$ is

$$\frac{y_1 - 2d}{x_1 - b} \cdot \frac{y_1}{x_1 - 2b} = -1 \Leftrightarrow 2dy_1 - y_1^2 = x_1^2 - 3bx_1 + 2b^2$$

$$\Leftrightarrow 2dy_1 = x_1^2 + y_1^2 - 3bx_1 + 2b^2.$$

Consider the equality

$$2dy_1 = x_1^2 + y_1^2 - 3bx_1 + 2b^2 \Leftrightarrow 2dy_1 = DH^2 - 3DH^2 + 2(DH^2 + HC^2)$$

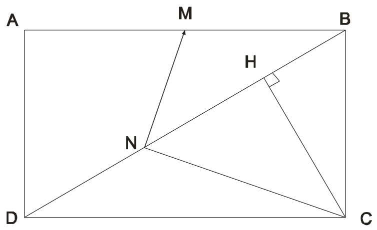
$$\Leftrightarrow 2dy_1 = -2DH^2 + 2(DH^2 + HC^2)$$

$$\Leftrightarrow \ dy_1 = HC^2 \Leftrightarrow dy_1 = HD \, . \, HB$$

$$\Leftrightarrow \ \frac{y_1}{HD} = \frac{HB}{BC} \Leftrightarrow \cos ADB = \cos HBC.$$

This is obvious. Thus, we have $MN \perp NC$, which is $CNM = 90^0$.

***Solution 6** (The transformative method)*



Considering the vectorial rotation -90^0, we have

$$\overrightarrow{DA} \mapsto \overrightarrow{DA'} = x \, . \, \overrightarrow{DC}$$

$$\overrightarrow{HB} \mapsto \overrightarrow{HC'} = y \, . \, \overrightarrow{HC}.$$

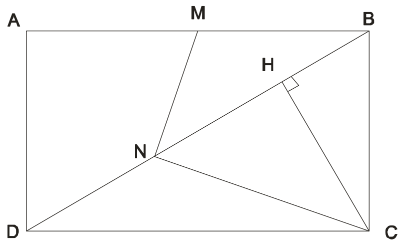
Since $\dfrac{HB}{HC} = \dfrac{DA}{DC} \Rightarrow x = y = k.$

Thus

$$\overrightarrow{NM} = \frac{1}{2}(\overrightarrow{DA} + \overrightarrow{HB}) \mapsto \overrightarrow{NM'} = \frac{1}{2}k(\overrightarrow{DC} + \overrightarrow{HC}) = k\overrightarrow{NC}.$$

Hence $MN \perp NC,$ which is $CNM = 90^0$.

***Solution 7** (The complex method)*



Suppose that $A(a), B(b), C(c), D(d), M(m), N(n), H(h)$.

We have $2m = a + b$; $2n = d + h$.

We need to prove $m - n = i(c - n)$

Or we need to prove $m - n = i(c - \dfrac{d+h}{2}) \Leftrightarrow 2(m - n) = i(2c - d - h)$.

We have $4(m - n) = 2(2m - 2n) = 2(a + b - d - h)$.

Thus, the thing which needs to be proved is equivalent to

$$2(a + b - d - h) = 4ic - 2i(d + h).$$

By the hypothesis, $ABCD$ is a rectangle and $CH \perp BD$, so we have

$$b - h = i(c - h) \Rightarrow b - h = ic - ih \Rightarrow h = \frac{(b - ic)(1 + i)}{1 - i^2} = \frac{b + c + i(b - c)}{2}$$

$$\Rightarrow 2h = b + c + i(b - c).$$

The thing which needs to be proved is equivalent to

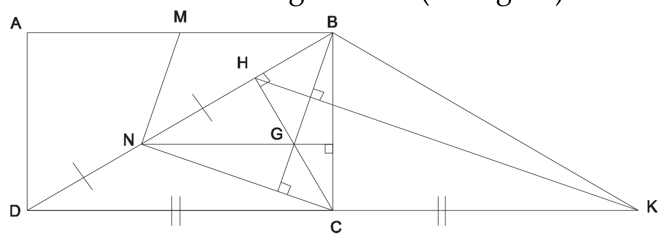
$$\begin{aligned} 2(a + b - d) - 2h &= 4ic - 2id - 2ih \\ \Leftrightarrow 2(a + b - d) - 4ic + 2id &= 2h(1 - i) \\ \Leftrightarrow 2(a + b - d) - 4ic + 2id &= \big(b + c + i(b - c)\big)(1 - i) \\ \Leftrightarrow 2a + 2b + 2d(i - 1) - 4ic &= (b + c)(1 - i) + (i + 1)(b - c) \\ &= b + c - ib - ic + ib - ic + b - c \end{aligned}$$

Or we need to prove that

$$a + d(i - 1) - ic = 0 \Leftrightarrow a - d = i(c - d).$$

This is obvious. Thus, we have $m - n = i(c - n)$, which is $MN \perp NC$, or $CNM = 90^0$.

By drawing byroads, we obtain the similar problems of the problem 2. If we take the point K on the opposite ray of the ray CD such that C is the midpoint of CK, then CN is the midline of the triangle DHK (the figure).



Thus, $NC \,//\, KH$.

By the proof 1 of the problem 2, we have $BG \perp NC$.

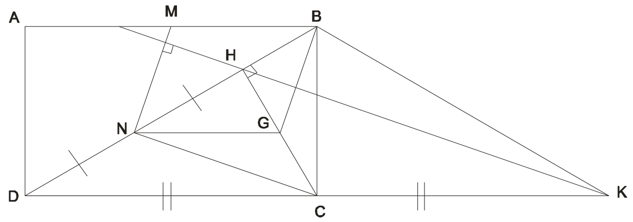
From two these things, we have $KH \perp BG$.

Thus, we have just proved the similar problem of the problem 2 as follow

Problem 3

Given a triang1e BCD with $C = 90^0$; the altitude CH. Let G be the midpoint of CH. Let K be the point symmetric to D with respect to the point C. Prove that $KH \perp BG$.

Combining the problem 2 with the problem 3, we see that $KH \perp BG$. On the other hand $BG \,//\, NM$. Thus, $KH \perp MN$.

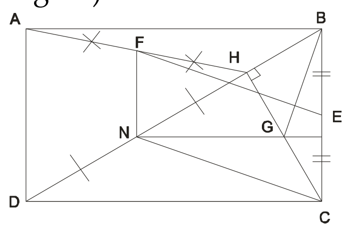


We obtain the following problem

Problem 4

$ABCD$ is a rectangle. Let CH be the altitude of the triangle BCD. Let M be the midpoint of AB, N be the midpoint of DH. Let K be the point symmetric to D with respect to the point C. Prove that $KH \perp MN$.

Using the parallel lines to AM or BN, we obtain problems which are similar to the problem 2. Connect AH. Let E be the midpoint of segment BC, F be the midpoint of segment AH (the figure).

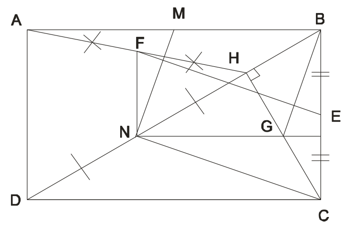


We have $CNFE$ being a parallelogram, so $EF // CN$. Because $CN \perp BG$, $EF \perp BG$. Thus, we have just proved the similar problem of the problem 2 as follow

Problem 5

$ABCD$ is a rectangle. Let H be the projection from C onto BD. Let G, E, F be the midpoints of segments CH, BC and AH, respectively. Prove that $EF \perp BG$.

We now combine the problem 2 and the problem 4, then we see that $NM // BG$ and $BG \perp EF$.



From this, we have the new following problem

Problem 6

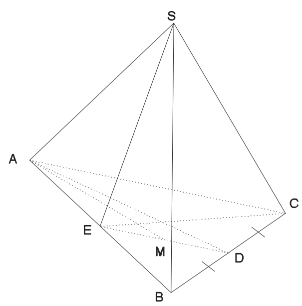
$ABCD$ is a rectangle. Let H be the projection from C onto BD. Let M, N, E, F be the midpoints of AB, DH, CB, AH, respectively. Prove that $MN \perp EF$.

From the problem 2, we generalize it to the problem in the space as follow

Problem 7

$SABC$ is a pyramid with ABC is isosceles at A. Let D be the midpoint of segment BC. Draw DE such that $DE \perp AB \ (E \in AB)$. Know that $SE \perp (ABC)$. Let M be the midpoint of DE. Prove that $AM \perp (SEC)$.

Indeed, we have $SE \perp (ABC)$, so $SE \perp AM$.



By the problem 2, $AM \perp CE$.

From two results, we have $\left.\begin{matrix} AM \perp SE \\ AM \perp CE \end{matrix}\right\} \Rightarrow AM \perp (SEC)$.

A generalization of problem 2 is as follows
Problem 8
$ABCD$ is a parallelogram. Let H be the projection from C onto BD. Take the points M on AB, N on HD and K on HC such that $\frac{HN}{HD} = \frac{HK}{HC} = \frac{BM}{BA}$. Prove that $MN // BK$.

The beauty in mathematical formulas is that mathematical results in different areas are connected, which is hard to realize at the very beginning. This connection is described as deep.

The example for the previous statement is the following Euler's identity: $e^{i\pi} + 1 = 0$.

Physicist Richard Fetnman has regarded this as "our jewel" and "the most remarkable formula in mathematics".

2.4.2. Categorizing mathematical beauty into inner and outer beauty

Outer mathematical beauty is the visual feature that affects a person's senses. A drawing, a formula, or a problem interests a person and makes him pay more attention. This is the outer mathematical beauty.

In contrast to outer beauty, there is inner mathematical beauty. It is impossible to see this beauty at first glance. The person has to spend a large amount of time contemplating, thinking, and studying in order to discover the inner connection between things, as well as the outer connection. When he has discovered these results, he feels happy and satisfied.

Both inner beauty and outer mathematical beauty are important. However, the inner beauty is harder to see, and a person has to have adequate ability to do so. In many cases, the discovery of the outer and inner beauty of a mathematical problem is synonymous to mathematical creativity.

For example, Fermat's Last Theorem: *Prove that the Diophantine equation* $x^n + y^n = z^n$ *has no integer solutions for* $n > 2$ and $x, y, z \neq 0$, the outer beauty is the simplicity of the equation, and the inner beauty is that it is an interesting and surprising theorem about the combination of integers in a formula. These integers are dancing harmoniously in the musical piece that is the formula, and this is the true beauty of Fermat's Last Theorem, expressed by mathematical symbols.

2.4.3. Categorizing mathematical beauty into different fields

a) Mathematical beauty in Computer Science

There is a close connection between Mathematics and Computer science. There are two applications of Mathematics in Computer Science. The first one is the mathematical theories models that are the basis for the development of Computer Science. The second one is using Mathematics to solve Computer Science problems and applications, finding mathematical theories and tools and putting them into use. Mathematics makes Computer Science more beautiful and profound. Most problems in Computer Science need the use of high to very high level modern Mathematics.

An example of mathematical beauty in Computer Science is the following algorithm

Problem 8

Write code that sums according to the expression $S = 1 + 2 + 3 + ... + (n-1) + n$.

The algorithm for this problem is:

1. $S = 0$; $i = 0$.
2. Input natural number n
3. While ($i \leq n$)
    3.1. Increment i by 1
    3.2. $S = S + i$.
4. Repeat from step 2
5. End algorithm

However, for this problem, we can use Mathematics to produce a result much faster. We have $1 + 2 + 3 + ... + (n-1) + n = \frac{n(n+1)}{2}$. So the algorithm can be:

1. Input natural number n.
2. Output $\frac{n(n+1)}{2}$.

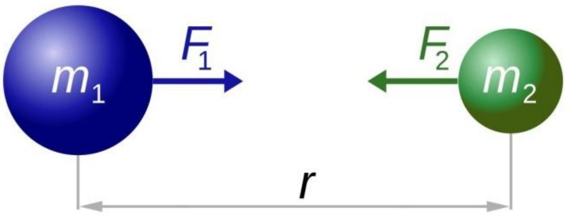
Above is only one example of mathematical beauty in Computer Science. Using Mathematics, one can simplify a great number of programming problems. This illustrates the close link between the two fields. Mathematics makes Computer Science more beautiful.

b) Mathematical beauty in Physics

mathematics and Physics are closely tied to each other. Without Mathematics, Physics wouldn't have developed so rapidly. Many physicists have built their theories on mathematical background. A typical example is Albert Einstein, who built his General Theory of Relativity based on mathematical background and non-Euclidean geometry. There is an entire subject called Equations of Mathematical Physics for students studying Physics.

Einstein once remarked that, *"beautiful theories"* are often accepted more readily, even if they have yet to be proven. An example is one of his own, most famous equation, $E = mc^2$. In a lecture at Oxford University in 1933, Einstein said that mathematical beauty was what guided him as a theoretical physicist. In other words, finding the simplest, most mathematically correct relationships, and then applying theories about how they operate. According to Einstein, the pinnacle of science is beauty and simplicity.

Newton's laws can be expressed in the form of the following equation:



$$F_1 = F_2 = G\frac{m_1 \times m_2}{r^2}$$

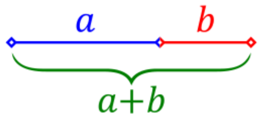
Beauty is eternal. So are beautiful equations. They are always true as they reflect what is inherent in nature, although previously hidden. Everything has its own law, which can be expressed in equation form and is comprehensible. One just needs to spend time looking into it, like Einstein said "Look deep into nature, and then you will understand everything better". (Cesti)

c) Mathematical beauty in interior design and in everyday life

Geometric beauty can be observed in many aspects of life. An example of this is ratios which are considered harmonious. A ratio in mathematics is a relationship between measurements of different things or different parts in one thing. For instance, the ratio between body measurements of someone who is 1.7m tall with a 90cm chest, 60cm waist and 90cm hips is 170:90:60:90, which is equal to 17:9:6:9. If one wants to make a 17cm tall figurine looking exactly like that person (or in mathematical terms, the figurine is geometrically similar to that person), the bust-waist-hips measurements of the figurine must be 9cm, 6cm and 9cm respectively, which are the real person's measurements divided by 10. (Nguyen, T., D)

Homothety, as well as the Thales' theorem is directly related to ratio and similarity. Homothety preserves ratio and maps a straight line into a straight line parallel to it. A cinema projector actually uses homothety to project films onto a big screen.

While mentioning ratio, it is crucial not to leave out the golden ratio since it appears in patterns in nature and plays an important role in human society.



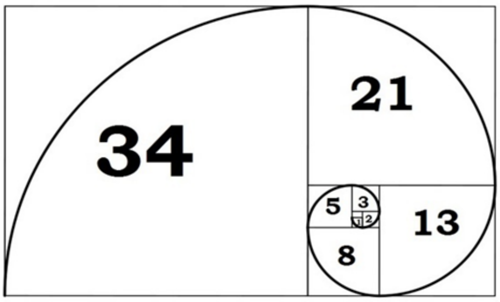
$a+b$ is to a as a is to b

$$\frac{a+b}{a} = \frac{a}{b} = \varphi = \frac{1+\sqrt{5}}{2} = 1.618$$

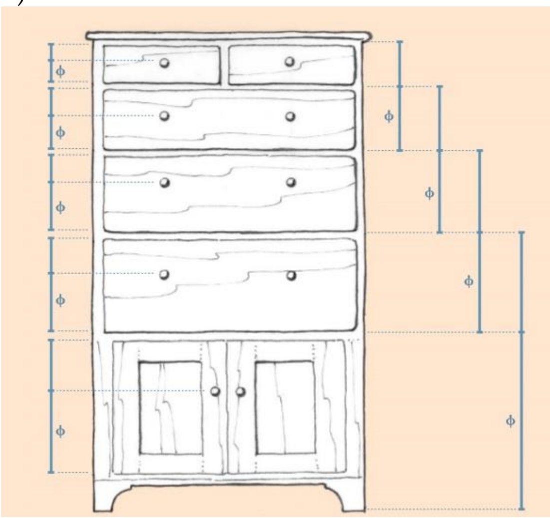
Consider two segments, a is the length of the longer segment, b is the length of the shorter segment and $a + b$ is the sum of a and b. When these quantities satisfy $\frac{a+b}{a} = \frac{a}{b}$, the ratio $\frac{a}{b}$ is said to be the golden ratio. Solving a quadratic equation gives the value of the ratio, which is 1.61803398875 (approximately 1.62). The Greek letter phi (φ) is used to represent the golden ratio.

Now, consider a golden rectangle (the ratio of the longer side to the shorter side is φ), there's some kind of connection to the natural essence in it. It appears that compositions displayed in a golden rectangle can make people feel at ease. They are also regarded as being well-organized and pleasing to the eyes.

Should the quantities a, b which satisfy the golden ratio be generally extended, one of them is the Fibonacci sequence. The Fibonacci sequence is defined by the recurrent relation $F_n = F_{n-1} + F_{n-2}$ with $F_1 = F_2 = 1, n \in N^*$. This sequence is of great importance because it represents numerous laws of nature. Arranging rectangles based on the Fibonacci numbers in ascending order results in the image of a spiral depicting the sequence - the golden spiral. The golden spiral occurs a lot in nature.



In interior design, the use of the golden ratio mainly focusing on golden rectangle can create spatial harmony. This ratio helps to design furnishings by keeping their widths and lengths in proportion. Furthermore, it suggests which part of the room should be decorated, which should be used to store the furniture, etc... (Ahd)



d) Mathematical beauty in poetry

The four lines of this poem is very known:

A Book of Verses undernearth the Bough,
A Jug of Wine, A Loaf of Bread – and Thou
Beside me singing in the Wilderness –
Oh, Wilderness were Paradise enow!

The four-line stanza above is a poem written by Omar Khayyam in Persian in the XI-XII centuries and was translated into English by Edward Fitzgerald (1809-1883) in the IX century. Of the millions of people who know Khayyam's poems, only a few know that he was a brilliant mathematician and astronomer in his time. In 1070, when he was only 22, Khayyam wrote a notable mathematical book named *Treatise on Demonstrations of Problems of Algebra*. In this book, "Pascal's triangle" (a triangular array of Newton's binomial coefficients) and a geometric solution to cubic equations – the intersection of a hyperbola with a circle - were found. Khayyam also contributed greatly to non-Euclidean geometry with a book titled *Explanations of the Difficulties in the Postulates of Euclid*. In the book, he proved some non-Euclidean properties of figures (though it is unknown whether or not non-Euclidean spaces really existed).

In Persia, Omar Khayyam originally achieved fame in the role of an astronomer. He was the one who introduced detailed astronomical tables (or ephemeris, which gives the positions of naturally occurring astronomical objects) and calculated the precise length of a solar year (365,24219858156 days). Based on these calculations, Khayyam proposed the Jalali calendar. The Jalali calendar is even more accurate than the present calendar.

People who've always seen mathematicians as impassive, unemotional people might be surprised if they find these sayings of none other than the "dry" mathematicians themselves:

"A mathematician who is not also something of a poet will never be a complete mathematician." - Karl Weierstrass.

"It is impossible to be a mathematician without being a poet in soul." - Sofia Kovalevskaya.

But why do mathematicians need to be "poets in soul"? It's simply because Mathematics is in accordance with poetry. The ultimate aim of both Mathematics and poetry is creating high aesthetic values. Therefore, only beautiful poems can last for a long time. The same goes for Mathematics; only beautiful mathematical works with high value can withstand the power of time and become classics. As Godfrey H. Hardy (1877-1947) once said: "*Beauty is the first test: there is no permanent place in the world for ugly mathematics*".

Both Mathematics and poetry are symbols of creativity. To create, one must have inspiration. If a "muse" is a poet's source of inspiration, a "maths' muse" must be the inspiration of mathematicians. Although they might serve different subjects on different occasions, "muse" or "maths' muse", they are in fact the same.

In Mathematics, not only can creativity result in new theorems, but also new areas of mathematics growing over time. It's no different in poetry, various poetic styles have been created through the course of history as old styles are not necessarily used.

Mathematics and poetry both require vivid imagination, perceptive creativity, language coherence, a thorough grasp of grammar and rules and so on. The language used in poetry is the normal language, while Mathematics has its own language with special concepts and symbols. However, they both use language to express ideas.

There's an especially significant quality which Mathematics and poetry share, that is *succinctness*. As British poet Robert Browning (1821 – 1889) once said: "*All poetry is putting the infinite within the finite*". Voltaire (1694 – 1778), a renowned philosopher also said: "*One merit of poetry few persons will deny: it says more and in fewer words than prose*". Mathematics, too, is succinct. The mathematical concepts and theorems can be very short, but comprehensive. It's as if they contain a whole universe in such few words and because of this, it's not always easy to understand Mathematics, or poetry. (Nguyen, T., D.)

e) Mathematical beauty in other fields

Mathematics has a tremendous impact on all life aspects nowadays, from natural environment to social life. For instance, thanks to simulation modeling, engineers can predict and solve many technical problems. Mathematics has undoubtedly become extremely important in the modern world.

6. Conclusion

Mathematical beauty is a relatively abstract concept. There's no one who can quantify or measure it. It is also highly subjective. Whether or not a mathematical problem is beautiful really depends on the perspective of the one who solves it. Some fundamental traits that mathematical beauty possesses are: repetition, symmetry, harmony, non-monotonicity and human-relatedness. There are various ideas of categorizing mathematical beauty. It can be categorized based on problem developing, problem solutions or mathematical formulae. Beauty can be on the inside or outside. But no matter how mathematical beauty is categorized, it's undeniable that Mathematics is truly beautiful and there needs to be more in-depth researches on the beauty of it.

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